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BRIEF COMMUNICATION

ADDED MASS COEFFICIENTS FOR UNIFORM ARRAYS

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BASIC THEORY

It can be established in several ways (Wallis 1989, 1992) that the averaged equations of motion of a uniform array and the surrounding ideal fluid, having one-dimensional motion in the x-direction with phase-average velocities v_2 and v_1, respectively, can be expressed as

$$\rho_1 \dot{v}_1 + \rho_1(\epsilon_1\beta - 1)(\dot{v}_1 - \dot{v}_2) = -\frac{\mathrm{d}p}{\mathrm{d}x} + \rho_1 g_1 \quad [1]$$

$$\rho_2 \dot{v}_2 - \rho_1(\epsilon_1/\epsilon_2)(\epsilon_1\beta - 1)(\dot{v}_1 - \dot{v}_2) = -\frac{\mathrm{d}p}{\mathrm{d}x} + \rho_2 g_2 + f_2 \quad [2]$$

ϵ_1 and ϵ_2 are the volume fractions of the phases and are subject to the constraint:

$$\epsilon_1 + \epsilon_2 = 1 \quad [3]$$

p is the macroscopic pressure, g_1 and g_2 are body force fields and f_2 is the external force per unit volume of phase 2, acting on phase 2. "β" is the "resistivity", which Rayleigh (1892) and others have shown to be given by

$$\beta = 1 + \frac{3}{2}\frac{\epsilon_2}{1 - \epsilon_2 + O(\epsilon_2^{10/3})} \quad [4]$$

for an isotropic regular array of spheres.

Similar equations can also be set up in vector form (Wallis 1992).

More sophisticated approaches are needed when there is convective acceleration, the pressure gradient is non-uniform, or the structure of the array evolves.

The combination $(\epsilon_1\beta - 1)$ has been called the "exertia" (Wallis 1989). If [4] is used, then

$$(\epsilon_1\beta - 1) = \frac{\epsilon_2}{2} + O(\epsilon_2^{13/3}) \quad [5]$$

A related quantity is the "polarizability" (Wallis 1993)

$$d_2 = (\beta - 1)\epsilon_1/\epsilon_2 \quad [6]$$

which becomes, on using [4]

$$d_2 = \tfrac{3}{2} + O(\epsilon_2^{10/3}) \quad [7]$$

We now consider several illustrative cases of [1] and [2].

Example A: $g_l = g_2 = dp/dx = 0$.

Motion is caused entirely by f_2. Using [1] and [2] we obtain

$$\epsilon_1 \dot{v}_1 = \epsilon_2 \dot{v}_2 \left(\frac{\epsilon_1\beta - 1}{\epsilon_2\beta}\right) \quad [8]$$

$$\dot{v}_2\left[\rho_2+\rho_1\left(\frac{\epsilon_1\beta-1}{\epsilon_2\beta}\right)\right]=f_2 \qquad [9]$$

The factor in parentheses in [8] represents the ratio of the volumetric flux of fluid to the volumetric flux of particles, due to motion initiated by the particles, starting from rest. In the same sense as the "drift" discussed by Darwin (1952), it is the "added mass coefficient" (Wallis 1989).

$$C_W=\frac{\epsilon_1\beta-1}{\epsilon_2\beta} \qquad [10]$$

which also describes the "added inertia" in [9].

If [4] is used, [10] becomes

$$C_W=\tfrac{1}{2}(1-\tfrac{3}{2}\epsilon_2+\tfrac{3}{4}\epsilon_2^2-\tfrac{3}{8}\epsilon_2^3+O(\epsilon_2^{10/3})) \qquad [11]$$

Example B: $g_1=g_2=0$; constant volumetric flux

The constant volumetric flux condition is equivalent to the constraint

$$\epsilon_1\dot{v}_1+\epsilon_2\dot{v}_2=0 \qquad [12]$$

which leads, on using [1] and [2], to

$$\rho_1\dot{v}_2(\beta-1)=\frac{\mathrm{d}p}{\mathrm{d}x} \qquad [13]$$

$$\dot{v}_2\left[\rho_2+\rho_1\left(\frac{\beta-1}{\epsilon_2}-1\right)\right]=f_2 \qquad [14]$$

If desired, the term in parentheses in [14] could be called the "added mass coefficient" for this case (Zuber 1964)

$$C_Z=\frac{\beta-1}{\epsilon_2}-1 \qquad [15]$$

though there is no longer an equivalent interpretation for "drift" since [12] replaces [8].

If [4] is used we obtain

$$C_Z=\tfrac{1}{2}+\tfrac{3}{2}[\epsilon_2+\epsilon_2^2+\epsilon_2^3+O(\epsilon_2^{10/3})] \qquad [16]$$

Example C: $g_1=g_2=0$, $v_2=0$

In this case the array is held stationary while fluid accelerates past it. The resulting pressure gradient and force density on the particles are

$$\rho_1\epsilon_1\beta\dot{v}_1=-\frac{\mathrm{d}p}{\mathrm{d}x} \qquad [17]$$

$$-\rho_1\dot{v}_1\frac{\epsilon_1}{\epsilon_2}(\beta-1)=f_2 \qquad [18]$$

If [4] is used in [18], we obtain

$$-\rho_1\dot{v}_1[\tfrac{3}{2}+O(\epsilon_2^{10/3})]=f_2 \qquad [19]$$

which is remarkably simple.

The factor appearing in [18] and [19] is the same as d_2 in [6], which could have been derived by considering how the dipole moment of the array "resists" the applied fluid motion.

Example D: $f_2=0$; $\epsilon_1\dot{v}_1+\epsilon_2\dot{v}_2=0$

This example corresponds to the vertical acceleration of an array under gravity in a closed container moving at constant speed. The result is similar to example B;

$$\dot{v}_2\left[\rho_2+\rho_1\left(\frac{\beta-1}{\epsilon_2}-1\right)\right]=(\rho_2-\rho_1)g \qquad [20]$$

Other definitions of "added mass" may be derived, as for example when the pressure difference built up across the array is used to push fluid through an "external impedance" (Cai & Wallis 1993). Sometimes, as when the pressure gradient is the only term on the right hand side in [1] and [2], no clear definition emerges.

OTHER FORMS OF BASIC EQUATION

The combination $\epsilon_1 \times [1] + \epsilon_2 \times [2]$ yields the overall equation of motion for the mixture

$$\epsilon_1 \rho_1 \dot{v}_1 + \epsilon_2 \rho_2 \dot{v}_2 = -\frac{dp}{dx} + \epsilon_1 \rho_1 g_1 + \epsilon_2 \rho_2 g_2 + \epsilon_2 f_2 \qquad [21]$$

Whereas subtraction of [1] from [2] gives

$$\rho_2 \dot{v}_2 - \rho_1 \dot{v}_1 - \rho_1 \frac{\epsilon_1 \beta - 1}{\epsilon_2} (\dot{v}_1 - \dot{v}_2) = \rho_2 g_2 - \rho_1 g_1 + f_2 \qquad [22]$$

which appears like an "equation of motion for the array" in which there is a "hydrostatic" component, $\rho_1 \dot{v}_1$, due to the acceleration of the continuous phase.

The coefficients in the "inertial coupling" term in [1], [2] and [22] are all different and none of them is equal to C_W or C_Z. The use of them interchangeably as a single "added mass coefficient" is the cause of much mistaken identity in the technical literature.

EQUIVALENCE OF FORMULATIONS

All of the equations presented so far are mutually consistent and can be derived from each other. For instance, one can start from the particular solution given in one of the examples and impose a suitable uniform acceleration on it in order to create a more general equation such as [22]. There is nothing "more correct" about any of these equivalent formulations.

The commonest reason why the users of one formulation do not realize that it is equivalent to the others is that they write down only one of the two momentum balance equations and are therefore unable to transform it.

In a similar way, it is simple to take Zuber's (1964) example of the motion of a sphere inside a fluid-filled stationary sphere (corresponding to example B) and impose an acceleration on the entire field to obtain [1] and [2], with dp/dz corresponding to the force per unit volume on the external spherical shell. The condition of no restraining force on the outer sphere gives the result in example A (it is the same as the condition of uniform potential there (Wallis 1993)).

Various other transformations are possible, by introducing composites, such as the mean volumetric flux, or bulk velocity:

$$j = \epsilon_1 v_1 + \epsilon_2 v_2 \qquad [23]$$

and the "drift flux":

$$j_{12} = \epsilon_1 \epsilon_2 (v_1 - v_2) \qquad [24]$$

As in diffusion theory, the relative motion may be referred to the mean volumetric flux, e.g.

$$v_1 - j = \epsilon_2 (v_1 - v_2); \quad v_2 - j = -\epsilon_1 (v_1 - v_2) \qquad [25]$$

and [22] expressed as

$$\rho_2 \dot{v}_2 - \rho_1 \dot{j} + \rho_1 \left(\frac{\beta - 1}{\epsilon_2} - 1 \right) (\dot{v}_2 - \dot{j}) = \rho_2 g_2 - \rho_1 g_1 + f_2 \qquad [26]$$

which leads to the appearance of C_Z in the coupling term.

Alternatively, a "reference velocity" resembling an "effective" velocity of the continuous phase (as though ϵ_1 in [23] were modified by a factor β) may be defined

$$U = \epsilon_1 \beta v_1 + (1 - \epsilon_1 \beta) v_2 \qquad [27]$$

and used to convert [1] and [2] to the simpler forms

$$\rho_1 \dot{U} = -\frac{dp}{dx} + \rho_1 g_1 \qquad [28]$$

and

$$\rho_2 \dot{v}_2 + \rho_1(\epsilon_1/\epsilon_2)(\dot{v}_1 - \dot{U}) = -\frac{dp}{dx} + \rho_2 g_2 + f_2 \qquad [29]$$

while [22] can be rearranged to

$$\rho_2 \dot{v}_2 - \rho_1 \dot{U} + \rho_1 \frac{\epsilon_1 \beta - 1}{\epsilon_2 \beta}(\dot{v}_2 - \dot{U}) = \rho_2 g_2 - \rho_1 g_1 + f_2 \qquad [30]$$

with C_W appearing as the coefficient in the coupling term.

In works by Wallis (1989, 1991) and Geurst (1985, 1986) the reference velocity that appears in [27] is equal to minus the gradient of the "macroscopic potential"

$$-\nabla\Phi = \mathbf{v}_1 + (\epsilon_1 \beta - 1)(\mathbf{v}_1 - \mathbf{v}_2) \qquad [31]$$

which is related to the bulk velocity by

$$-\nabla\Phi = \mathbf{j} + d_2 \epsilon_2(\mathbf{v}_1 - \mathbf{v}_2) \qquad [32]$$

where the second term in [32] is the dipole moment per unit volume in response to the relative motion. If the array is not isotropic, β and d_2 become tensors.

We may now eliminate $\dot{U}$ between [28] and [30] and use [10] to obtain

$$\dot{v}_2(\rho_2 + C_W \rho_1) = -\frac{dp}{dx}(1 + C_W) + \rho_2 g_2 + C_W \rho_1 g_1 + f_2 \qquad [33]$$

which resembles the equation of motion of a pseudo-fluid made up of unit volume of dispersed phase and a volume C_W of fluid entrained with it and moving at the same velocity. Indeed, [28] and [33] may be derived directly (Wallis 1989) from the Cook–Harlow (1984) idealized model in which a fraction $1/\beta$ of the volume contains fluid moving at speed U, while the rest of the fluid, occupying a volume fraction $\epsilon_1 - 1/\beta$, is entrained with the dispersed phase which occupies a volume fraction ϵ_2. Geurst (1988) exploits this idea in a more general form.

In view of the simplicity, in both form and interpretation, of [28], [29], [30] and [33], I prefer this version, but the choice is a matter of taste and does not invalidate any of the other equivalent formulations.

It is possible to eliminate $\dot{v}_2$ from [1] and [2] to obtain another "equation of motion for the fluid":

$$\rho_1 \dot{v}_1 = -\frac{dp}{dz} + \rho_1 g_1 + (\rho_2 g_2 + f_2 - \rho_1 g_1)\frac{\epsilon_2}{\epsilon_1}\frac{\epsilon_1 \beta - 1}{\beta - 1} \qquad [34]$$

The final factor in [34] is the ratio of the exertia to the polarizability.

Should other terms, describing further external or mutual forces of interaction, be added to [1] and [2], they will simply appear, multiplied by some factor, as additional terms in all of the equations presented here, without changing their form.

Related arguments may be found in Smereka & Milton (1991).

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